THE MINIMUM PROBLEM OF THE STABILITY OF EQUILIBRIUM OF A SOLID BODY PARTIALLY FILLED WITH A LIQUID

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A number of papers have considered the problem of the motion of a solid body with cavities partially filled with liquid (cf., for example [1-6]). The equations of motion were based upon the assumption of potential motion of the liquid and the usually assumed theory of waves of small amplitude and theory of small oscillations of a solid body. Following Poincaré, the displacement of the liquid from the equilibrium position was expanded in a certain series of functions with time-dependent coefficients, and the problem was reduced to an infinite set of equations of second order.

The problem of stability of equilibrium of a mechanical system acted upon by potential forces may be solved exactly with the help of a theorem of Lagrange on the stability of equilibrium, in which state the potential energy has an absolute minimum. The criterion for the minimum potential energy of a system with a finite number of degrees of freedom is well known and is the condition for positive-definiteness of the second variation of the potential energy. If a system has an infinite number of degrees of freedom, then the problem is considerably more complex. The expansion of the equation of the disturbed surface in a certain series of functions reduces the problem of minimization to a problem of the positive-definiteness of an infinite quadratic form, which is the result of taking the second variation of the potential energy. For the case of a heavy fluid it was shown [3] that this quadratic form is positive-definite when a certain quadratic form in the variations of the coordinates of the solid body is positive-definite. The latter quadratic form is the second variation of the potential energy of the rigid body and the frozen liquid, to which has been added a correction for the liquid contents. The correction for the liquid contents is a quadratic form in the variations of the coordinates of the solid body with coefficients which are sums of an infinite number of terms.

From the condition of positive-definiteness of the sum of these quadratic forms it would have been possible to carry out a rigorous derivation of the stability of equilibrium of the system in the sense of Liapunov; however, for application this sum had to be calculated, and this was done in [7] only for a particular case.

Below the problem of minimum potential energy of the system also reduces to a problem of minimizing a certain function of the coordinates of the solid body, consisting of the potential energy of the frozen system together with a nonpositive correction for the liquid contents. The expansion of this correction starts with the quadratic form in the variations of the coordinates of the solid body with coefficients in the form of double integrals of known functions over known regions.

1. We consider a vessel having the form of a closed rectangular parallelepiped free to rotate about a fixed axis y passing through the axis of symmetry of the vessel and perpendicular to its sides. If the heavy vessel is filled with a heavy liquid, then in equilibrium the walls of the vessel are vertical, and the free surface of the liquid is horizontal.

Let the straight line ab be the projection of the y-axis on the surface of the liquid in the equilibrium position (we will imagine it to be rigidly attached to the vessel) and let P be a plane rigidly fixed to the vessel and coincident with the free surface of the liquid in the equilibrium position. If the vessel is inclined at an angle θ to the vertical, then the potential energy of the liquid in this position will have a minimum in comparison with its other possible positions if its surface is the horizontal plane P_1; that is, the position of an incompressible liquid, bounded above by the plane P_1, will be possible if P_1 passes through the straight line ab, and consequently satisfies the condition of conservation of volume. We call this position of the system the position θ, P_1.

We put the system initially in the position θ, P, imagining that the liquid is frozen, and we let δU_1 be the variation of potential energy of the system for this displacement. Then we transfer the liquid located above the plane P_1 and below the plane P, so that it is below the plane P_1, and we obtain the position θ, P_1. The variation δU_2 in the potential energy for this displacement will obviously be negative. The total variation in the potential energy of the system is $\delta U^{1+2} = \delta U_1 + \delta U_2 < \delta U_1$ where δU^{1+2} depends only on the angle θ.

Consequently, in order that δU be positive, it is necessary and sufficient that δU^{1+2} be positive-definite. We note that for this it is necessary that δU_1 be unable to take negative values. Thus the problem of the minimum potential energy of the system is reduced to the minimization of δU^{1+2}, which depends only on the angle, and the solution of this problem presents no difficulty.

In the general case we imagine a solid body with a cavity partially filled with a homogeneous incompressible liquid of density σ.

Let the position of the solid body relative to a fixed system of coordinates O, x, y, z be given by scleronomic and holonomic generalized coordinates q_i $(i < 6)$, and let there act on it steady forces with potentials $U'(q_i)$. The volume forces acting on the liquid are also assumed to have a potential $W(x, y, z)\,\sigma\, d\tau$ for an element of volume $d\tau$.

When the system is in equilibrium, let the values of the q_i be zero, and let the liquid occupy a certain amount of a simply connected region D_k°, bounded by S, the surface of the cavity, and by sections of the surface $W = \alpha_k$.

Let Q_k be a certain point of the free surface of the liquid, M_k the geometric positions of the first points of intersection of continuous curves with origin at Q_k and lying on $W = \alpha_k$, with the surface S. The curve M_k will be called the boundary of the surface $W = \alpha_k$ and the volume of the liquid will be called V_k. We do not exclude from consideration the case where the simply connected region occupied by the liquid is bounded by the sides S, and by a surface $W = \alpha_k$ with a boundary consisting of several closed curves M_{kj}.

It is clear that in the equilibrium state the liquid volume does not contain points $W > \alpha_k$.

Let T_k be a certain point of the boundary M_k of the free surface, $n_1(T_k)$ a unit vector normal to the surface S at the point T_k and directed into the cavity, $n_2(T_k)$ the unit vector normal to the surface $W = \alpha_k$ directed to the side $\alpha > \alpha_k$, and $\theta(T_k)$ the angle between these normals measured from n_1 to n_2. We assume that the angle θ is bounded by certain constant limits $\pi > \theta_1 > \theta(T_k) > \theta_0 > 0$ for all points T_k and for any k. We assume that the vector n_1 varies continuously with the location of the points of the surface S in the neighborhood of the boundary M_k, and we suppose that the unit vector n_2 normal to the family of surfaces $W = \alpha$ varies continuously with respect to x, y, z in that same neighborhood. We also assume that none of the α_k is the largest of all the possible values which W can take on in the neighborhood of $W = \alpha_k$.

Let M_{kj} be a certain closed curve belonging to the boundary of the region for a certain value α_k and given by the equation

$$x = x_j(t, \alpha_k, \beta^\circ), \qquad y = y_j(t, \alpha_k, \beta^\circ), \qquad z = z_j(t, \alpha_k, \beta^\circ) \tag{1.1}$$

where $\Phi(x, y, z) = \beta^\circ$ is the equation of the surface of the cavity in the neighborhood of the boundary, t is a parameter varying between the limits $0 \leqslant t \leqslant l^\circ$, and the functions x_j, y_j, z_j are periodic in t with period l°. According to the assumptions made concerning the properties of the normal, the vector

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \\ \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \end{vmatrix} = \left(\frac{\partial W}{\partial y}\frac{\partial \Phi}{\partial z} - \frac{\partial W}{\partial z}\frac{\partial \Phi}{\partial y}\right)\mathbf{i} + \ldots$$

vanishes nowhere in a small region of the curve M_{kj}. Consequently, according to the existence theorems for implicit functions there exist functions

$$x = x_j(t, \alpha_k + \Delta\alpha, \beta^\circ, q_i), \qquad y = y_j(t, \alpha_k + \Delta\alpha, \beta^\circ, q_i) \tag{1.2}$$
$$z = z_j(t, \alpha_k + \Delta\alpha, \beta^\circ, q_i) \qquad (0 \leqslant t \leqslant l,\ i \leqslant 6)$$

which are solutions of the equations

$$W(x, y, z) = \alpha_k + \Delta\alpha, \qquad \Phi'(x, y, z, q_i) = \beta^\circ \tag{1.3}$$

where $\Phi' = \beta^\circ$ is the equation of the surface of the cavity when the solid body is displaced from its equilibrium position. When $\Sigma q_i^2 + \Delta\alpha^2 \to 0$, the functions (1.2) transform continuously into the functions (1.1). The continuity of these functions is uniform in t, hence they will also be periodic with the same period l°, and consequently the curve (1.2) is closed. This means that the region $D_k'(\Delta\alpha, q_i)$, bounded by the surface $W = \alpha_k + \Delta\alpha$ and the surface S, and which transforms continuously into the region D_k° as $\Sigma q_i^2 + \Delta\alpha^2 \to 0$, will also be simply connected and will contain only the points $W < \alpha_k + \Delta\alpha$.

If q_i varies in the closed region $r^2 = \Sigma q_i^2 \leqslant H^2$, where $H > 0$ is a sufficiently small constant, then for every α_k a constant $\gamma_k > \alpha_k$ can be found such that for any q_i taken from this region, the surface of the cavity S and the surfaces $W = \gamma_k$, $W = \alpha_k + \Delta\alpha$ will bound a closed region $C_k(\Delta\alpha, \gamma, q_i)$ which does not contain the point $W < \alpha_k + \Delta\alpha$.

Here, as well as in what follows, by the surface $W = \alpha_k + \Delta\alpha$ we will mean the portion of this surface which as $\Sigma q_i^2 + \Delta\alpha^2 \to 0$ transforms continuously into the free surface $W = \alpha_k$ of the liquid in the equilibrium position. The existence of the surface $W = \gamma_k$ follows from the previous considerations as long as γ_k is close to α_k. However, in many cases of practical importance γ_k may be taken sufficiently far from $\alpha_k + \Delta\alpha$, but if the region bounded by the surface S is closed, then it may turn out that any possible displacements of the liquid do not cross a certain surface $W = \gamma_k$, where the value γ_k depends only on the radius of the sphere $r^2 = H$. The surface γ_k, as we shall see below, plays a significant role in the determination of stability.

2. We now assume that for given q_i the liquid completely fills the region $D_k''(\Delta\alpha,\ q_i)$. It is not difficult to see that the potential energy of this liquid volume has a minimum with respect to all possible positions of the liquid for given q_i which do not exceed the limits of the region $C_k(\Delta\alpha,\ \gamma_k,\ q_i)$. Actually, when any particle of liquid passes across the surface $W = \alpha_k + \Delta\alpha$ it goes into the region $C_k(\Delta\alpha,\ \gamma_k,\ q_i)$ and as a consequence its potential energy is increased. The position of the fluid occupying the whole region $D_k''(\Delta\alpha,\ q_i)$ will be possible if the volume V_k'' of this region equals the volume V_k, i.e. the condition of incompressibility will be satisfied.

We now address ourselves to the determination of the $\Delta\alpha$ for which this condition will be satisfied.

If the liquid is frozen in the equilibrium position and then the frozen system is displaced, then the equation of the frozen surface for any position of the body will be $W(x',\ y',\ z') = \alpha_k$, where $O,\ x',\ y',\ z'$ is a system of rectangular coordinates attached to the solid body and coincident with the fixed system $O,\ x,\ y,\ z$ in the equilibrium position. In the following we will also call this surface the frozen surface and denote its boundary by M_k', while the region containing the frozen liquid we will call D_k .

We denote by $E_k(\Delta\alpha,\ q_i)$ the "difference" of the regions $D_k''(\Delta\alpha,\ q_i)$ and D_k', i.e. the totality of points belonging to D_k' and not belonging to $D_k''(\Delta\alpha,\ q_i)$, which make up the region F_k, and the totality of points belonging to D_k'' and not belonging to D_k', which constitute the region G_k.

As was shown above, the region F_k consists of the points

$$W\ (x',\ y',\ z') < \alpha_k, \qquad W\ (x,\ y,\ z) > \alpha_k + \Delta\alpha, \text{ inside } S$$

and the region G_k consists of the points

$$W\ (x',\ y',\ z') > \alpha_k, \qquad W\ (x,\ y,\ z) < \alpha_k + \Delta\alpha, \text{ inside } S$$

The region E_k is the "sum" of the regions G_k and F_k.

We now assume that there exists a one-to-one transformation which is continuously differentiable

$$x = x(\xi, \eta, \zeta), \qquad y = y(\xi, \eta, \zeta), \qquad z = z(\xi, \eta, \zeta)$$

in a certain neighborhood of the free surface $W = \alpha_k$, and that in this neighborhood the Jacobian of the transformation is

$$J = \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} > \varepsilon > 0$$

We also assume that $\partial W / \partial \zeta$ is bounded by the limits $0 < a < \partial W / \partial \zeta < b$ and is continuous in this same neighborhood, and consequently $1/a > \partial\zeta / \partial W > 1/b > 0$ in this neighborhood. The latter two limits are not of importance, but are introduced to simplify the proof.

The condition of conservation of volume takes the form

$$\delta V_k = \int\limits_{D_k''} d\tau - \int\limits_{D_k'} d\tau = \int\limits_{G_k} d\tau - \int\limits_{F_k} d\tau = 0 \tag{2.1}$$

Let ξ', η', ζ' be curvilinear coordinates connected rigidly to the solid body and transforming into ξ, η, ζ in the equilibrium position, and let the coordinates ξ', η' vary in the region P_{kj} in ranging over the portion of the frozen surface bounded by the closed curve M_{kj}' belonging to boundary M_k'.

Let the function $W(x', y', z')$ transform into the function $W'(\xi', \eta', \zeta')$ under the transformation to coordinates ξ', η', ζ', while the function $W(x, y, z)$ transforms into the function

$$W''(\xi', \eta', q_i) = W'(\xi', \eta', \zeta') + \sum_i \frac{\partial W''}{\partial q_i} q_i + O(r)$$

where $O(r)$ is a small quantity of higher order than r. Let ζ vary according to the equations $\zeta' = \Psi(\xi', \eta', \alpha_k + \Delta\alpha, q_i)$, $\zeta' = \Psi(\xi', \eta', \alpha_k, 0) = \Psi^0$ as the point moves along the surfaces $W'' = \alpha_k + \Delta\alpha$, $W' = \alpha_k$ respectively, where the right hand sides of the equations are single-valued, continuous in all the arguments, and differentiable with respect to α.

The condition of conservation of volume (2.1) may be written

$$\delta V_k = \sum_j \iint\limits_{P_{kj}} d\xi' d\eta' \int\limits_{\Psi^0}^{\Psi} J\, d\zeta' + O(r') \qquad \left(r'^2 = \sum q_i^2 + \Delta\alpha^2\right) \tag{2.2}$$

Here $O(r')$ represents the algebraic sum of the volumes adjoining on M_{kj}' and bounded by the following surfaces: the cylinder $\xi' = \xi_0'(t)$,

$\eta' = \eta_0'(t)$ passing through the curve M_{kj}', the surface S and the surface $\zeta' = \Psi$, where these volumes have a plus sign if they belong to the region G_k or are outside the region F_k, and a minus sign in the opposite case. The section of any such region cut by a plane R perpendicular to the contour M_{kj}' at the point T_k will be a curvilinear triangle T_kAB with the side T_kA lying on the cylinder, the side T_kB lying on the surface of the cavity, and the side AB lying on the surface $\zeta' = \Psi(W(x, y, z) = \alpha_k + \Delta\alpha)$. The height of this triangle dropped from the vertex T_k will vanish together with $\Delta\alpha$. The angle $B \to \theta$ as $r' \to 0$, and consequently for r' sufficiently small this angle lies between the limits $\pi > \theta_1 > B > \theta_0 > 0$.

As $r' \to 0$, the angle A approaches the angle AT_kL, where T_kL is the line of intersection of the plane R and the frozen surface. It is not difficult to see that this angle also lies between similar limits. Actually, on the frozen surface $1/a > \partial\zeta'/\partial W' > 1/b$, since it is clear that $\partial\zeta/\partial W$ on the free surface of equilibrium is identical to $\partial\zeta'/\partial W'$ on the frozen surface. This means that the angle φ between the curve $\xi = \xi_0$, $\eta = \eta_0$, directed towards the side $\alpha > \alpha_k$, and the frozen surface at all points exceeds a certain limit $\varphi_0 > 0$. From the continuity of $\partial\zeta'/\partial W'$ it follows that the angle AT_kL also lies between the limits $\pi > \varphi_1 > \angle AT_kL > \varphi_2 > 0$. The above considerations show that the area of the triangle T_kAB is of order r'^2 and is of the same order as the algebraic sum of the volumes of the regions under consideration.

Introducing in place of ζ the variable $\mu = W'' - \alpha_k$, we rewrite equation (2.2) as follows:

$$\delta V_k = \sum_j \iint\limits_{P_{kj}} d\xi' d\eta' \int\limits_{\Delta\alpha}^{v_k} J \frac{\partial\zeta'}{\partial W''} d\mu + O(r') = 0 \qquad \Big(v_k = \sum_j \frac{\partial W''^\circ}{\partial q_i} q_i + O(r)\Big) \quad (2.3)$$

where the partial derivative $\partial W''^\circ/\partial q_i$ is calculated on the frozen surface $\varphi' = \Psi^\circ$. Noting that

$$\frac{\partial\zeta'}{\partial W''} = \frac{\partial\zeta'^\circ}{\partial W''} + O(r' + |\mu|), \qquad J = J^\circ + O(r' + |\mu|)$$

we rewrite (2.3) in the form

$$\delta V_k = \Delta\alpha \sum_j \iint\limits_{P_{kj}} J^\circ \frac{\partial\zeta'^\circ}{\partial W''} d\xi' d\eta' - \sum_j \iint\limits_{P_{kj}} J^\circ \frac{\partial\zeta'^\circ}{\partial W''} \sum_i \frac{\partial W''}{\partial q_i} q_i \, d\xi' \, d\eta' + O(r') =$$

$$= \nu_k \Delta\alpha - \sum_i \lambda_{kj} q_i + O(r') = 0$$

Since $\nu_k > 0$, this equation has the unique solution

$$\Delta\alpha_k = \frac{1}{\nu_k}\sum_j \lambda_{ki} q_i + O(r') \tag{2.4}$$

Thus for every k there exists a $\Delta\alpha_k(q_i)$ such that for any $r < H$, there is a possible position of the liquid, completely filling the region $D_k''(\Delta\alpha_k, q_i)$, for which the potential energy of the liquid attains a minimum in comparison with all other possible positions within the region $C_k + D_k''$.

3. The potential energy of the system will have in this sense a minimum in the equilibrium position if the function

$$U_{\min} = U'(q_i) + \sigma \sum \int_{D_k''} W\, d\tau$$

which depends only on q_i has a minimum. It may be represented in the form

$$U_{\min} = U' + \sigma\sum_k \int_{D_k'} W\,d\tau + \Big[\sigma\sum_k \int_{D_k''} W\,d\tau - \sigma\sum_k \int_{D_k'} W\,d\tau\Big]$$

We denote the sum of the first two terms, representing the potential energy of the frozen system, by U_1 and the remainder, enclosed in brackets, by U_2. We note that by hypothesis U_2 is never positive. The second variation of U_1 may be found by the usual method, and therefore we address ourselves to the calculation of U_2; we have

$$U_2 = \sum_k U_{2k}$$

$$U_{2k} = \sigma\int_{D_k''} W\,d\tau - \sigma\int_{D_k'} W\,d\tau = \sigma\int_{G_k}(\alpha_k+\mu)\,d\tau - \sigma\int_{F_k}(\alpha_k+\mu)\,d\tau =$$

$$= \sigma\int_{G_k}\mu\,d\tau - \sigma\int_{F_k}\mu\,d\tau \tag{3.1}$$

The last equality was written on the basis of the condition of conservation of volume (2.1). Making further transformations as in Section 2, we obtain

$$U_{2k} = \sigma\sum_j \iint_{P_{kj}} d\xi'\,d\eta' \int_{v_k}^{\Delta\alpha_k} \mu J \frac{\partial\zeta'}{\partial W''}\,d\mu - O(r'^2) =$$

$$= \frac{\sigma}{2}\sum_j \iint_{P_{kj}} (\Delta\alpha_k^2 - v_k^2)\, J^\circ \frac{\partial\zeta'^\circ}{\partial W''}\,d\xi'\,d\eta' + O(r^2) =$$

$$= \frac{\sigma}{2\nu_k}\left(\sum_i \lambda_j q_i\right)^2 - \frac{\sigma}{2}\sum_j \iint_{P_{kj}} \left(\sum_j \frac{\partial W''^\circ}{\partial q_i} q_i\right)^2 J^\circ \frac{\partial \zeta'^\circ}{\partial W''} d\xi' \, d\eta' + O(r^2)$$

Using the well-known inequality

$$\int f^2 d\tau \geqslant \left(\int f d\tau\right)^2$$

one may convince oneself that the sum of the second order terms may not take on positive values. However, it may happen that U_1 does not depend on all the coordinates. For example, let $U_1(q_1)$ depend only on one coordinate and let W'' have the form

$$W'' = W'(\xi', \eta', \zeta') + \frac{\partial W''(\xi', \eta', \zeta', 0, q_2 \ldots)}{\partial q_1} q_1 + O(q_1)$$

where $O(q_1)/q_1 \to 0$ as $q_1 \to 0$ uniformly for all ξ', η', ζ' in the neighborhood of the frozen surface and for all permissible q_2, q_3, ... Then repeating the previous arguments, we obtain

$$U_{2k} = \frac{\sigma}{2\nu} \lambda_1{}^2 (q_2, \ldots) q_1{}^2 - \frac{\sigma}{2}\sum_j \iint_{P_{kj}} \left(\frac{\partial W''^\circ(0, q_2, \ldots)}{\partial q_1}\right)^2 q_1{}^2 d\xi' \, d\eta' + O(r'^2) =$$
$$= \frac{\sigma}{2} N(q_2, q_3 \ldots) q_1{}^2$$

If it turns out that

$$\left(\frac{\partial^2 U_1}{\partial q_1{}^2}\right)^\circ + \frac{\sigma}{2} N(q_2, q_3 \ldots) > 0$$

for any q_2, q_3, ..., then the potential energy of the system will have a minimum with respect to q_1 and for arbitrary disturbances of the liquid which do not cross the surface $W = \gamma_k$. This case occurs in the problem of the stability of equilibrium of a pendulum filled with a heavy liquid.

It is necessary to note also the following circumstance. All of the double integrals above depend only on the form of the free surface in the position of equilibrium, i.e. the form of the function W and the form of the curve M_k of intersection of the free surface and the walls of the cavity, and they are independent of the form of the surface of the cavity in the neighborhood of this curve. This means that the second variation of the potential energy of the system does not change if the shape of the cavity changes, so that the potential energy of the frozen system and the curve M_k remain as before. The second variation of the potential energy depends only on the volume and surface integrals, hence it does not change sign if the cavity S is replaced by the cavity S', provided the volume consisting of the points of S' not belonging to S

and the volume consisting of the points S not belonging to S' are sufficiently small, and the whole region D_k° transforms continuously into a simply connected region. This is also true if the volume V_k of the liquid changes by little and the surface of the cavity is subject to the condition that the "projection" of the boundary M_k on the "plane" $\zeta = \zeta_0$ varies by a sufficiently small amount, i.e. the area bounded by the old and new projections is little changed. This means that if some stability criteria are obtained from the sign of the second variation, then they will be valid for variations in the shape of the cavity and the quantity of the liquid which are small in the above sense.

4. Following Liapunov [8], we introduce the following definition. We assume that the liquid belonging to a certain region D_k° in the equilibrium state is perturbed, and we consider the surface of the liquid at an arbitrary instant during its perturbed motion. From any point of this surface we imagine a pencil of straight lines joining it with all points of the surface of the equilibrium volume V_k, and from the segments of these lines between the chosen point of the liquid surface and the equilibrium surface we choose the smallest. By the equilibrium surface we mean the free surface and the walls of the cavity wetted by the liquid. Considering all points of the disturbed surface of the liquid, we choose the largest of these segments. We denote this positive quantity by N and call it the distance of the surface of the liquid from the equilibrium surface.

As before let V_k be the volume of the liquid and let q_k be the volume all points of which belong to both of the two regions, one bounded by the surface of the liquid and the other by the equilibrium surface. We call the difference $V_k - q_k$ the displacement of the liquid from the equilibrium state and denote it by Δ. If all possible surfaces of the liquid are considered for which the distances from the equilibrium surface are equal to a given quantity N, then it is clear that for these surfaces Δ may take on all values between zero and a certain limit depending on N which for $N = 0$ reduces to zero. Any single-valued and continuous function of N taking on only those values that Δ may attain for the same N and reducing to zero for $N = 0$ we will call a possible displacement and denote $\varphi(N)$. We assume also that the motion of liquid is such that N and Δ are continuous functions of the time t.

Definition. We impart to the system arbitrary displacements and velocities and consider the subsequent perturbed motion. If $\Sigma q_{i0}^2 = r_0^2$, the initial distance N_0 from the equilibrium surface and the initial magnitude of the kinetic energy T_0 may be chosen sufficiently small for all possible values of the rest of the initial data, so that r, the distance N of the liquid surface from the equilibrium surface and the kinetic energy T remain less than certain prescribed limits, no matter how small, for all times during the motion or at least until the displacement of the liquid from the equilibrium position does not become less than a certain prescribed possible displacement, no matter how small, then the equilibrium state under consideration is stable.

All possible initial data must be subject to the condition that the functions N and Δ corresponding to these quantities must be continuous functions of t during the entire motion.

For our problem, however, it will be convenient to consider the distance N and displacement Δ of the perturbed surface, not from the equilibrium surface, but from the boundaries of the region D_k''.

We consider now the region $C_k(\gamma_k, \Delta\alpha_k, q_i)$

$$W \leqslant \gamma_k, \quad W \geqslant \alpha_k + \Delta\alpha_k \quad \text{within } S$$

and treat n_1 as the minimum distance of a point on the surface $W = \gamma_k$ from the boundaries of the region D''. Varying r within the limits $r^2 < H^2$, where $H > 0$ is some small constant, we find N_k, the minimum value of n_1. We consider any possible displacement $\varphi_k(N)$, and from all possible positions of the system satisfying the inequalities $\Delta \geqslant \varphi(N)$, $N < N_k$, $r \leqslant H$, we choose that position for which U attains its smallest possible value $U_{N_k} > 0$.

We choose the initial conditions so that the following inequalities are satisfied:

$$T_0 + U_0 < U_{N_k}, \qquad \Delta_0 \geqslant \varphi_k(N_0)$$

If the energy of the system is conserved or if energy is dissipated, then

$$0 < U < U_{N_k} \quad \text{for } \Delta \geqslant \varphi_k(N)$$

Actually, if U becomes zero, then before that happens N, which varies continuously, must become equal to N_k. This may occur only if the inequality $\Delta \geqslant \varphi_k(N)$ is violated before U becomes zero, since $U > 0$ for all $\Delta \geqslant \varphi_k(N)$.

If however the inequality $\Delta \geqslant \varphi_k(N)$ is not violated, then neither is the inequality $0 < U < U_{N_k}$.

If the last inequality is satisfied, then the inequality

$$N < N_k, \quad r \leqslant H, T < U_{N_k}$$

will be satisfied also.

Thus we come to the conclusion that in order that the system be stable in the above-mentioned sense, it is sufficient that $U_{\min}$ be a positive-definite function of q_i. This proof, with certain differences in detail, was carried out by Rumiantsev [9].

Up to now we have been concerned with a cavity with one simply-connected volume D_k°. If there are several such volumes, then if all of the quantities having index k are understood to refer to the kth simply-connected volume, and T is understood to be the kinetic energy of the system, then by a similar argument we obtain the analogous results.

We continue the discussion, considering D_k° to be a single simply-connected volume.

If the cavity and the function W are such that the region, lying within the cavity and outside the region D_k'', into which a liquid particle may pass from the region D_k'' without crossing the walls of the cavity, does not contain the points $W < \alpha_k + \Delta\alpha_k$, then the position of the liquid completely filling the region D_k'', attains the minimum potential energy of the liquid for given q_i in comparison with all attainable displacements of the perturbed surface. In this case a more definite criterion regarding the stability of the liquid may be obtained. This definition also follows Liapunov [8]. However, he did not make use of this definition, since in using only one energy integral, stability in the sense of this definition can be shown only for one of the problems considered by him – the problem of the stability of the spherical equilibrium shape of a liquid mass under the influence of gravity.

Definition. If for any positive U, σ', δ less than certain limits there can be found limits U_0, σ_0', δ_0 such that for arbitrary initial values of r_0, the kinetic energy T_0, and the displacement Δ_0 subject to the inequalities

$$r_0 < H_0, \qquad T_0 < \sigma'_0, \qquad \Delta_0 < \delta_0$$

the inequalities

$$r < H, \qquad T < \sigma', \qquad \Delta < \delta$$

will be satisfied for all times during the motion, then this equilibrium state of the system is stable. We will call this stability with respect to the displacement r and T.

Let $U_{\min}^{\circ}$ be the minimum value of $U_{\min}$ on the sphere $r = H$ and let $d_k > 0$ be a constant such that for $\Delta > \delta_k$, $r \leqslant H$ the inequality $U > U_{\min}^{\circ}$

is satisfied.

If the system conserves energy or if energy is dissipated, and if T_0, r_0 and Δ_0 are chosen so small that the inequality

$$T_0 + U_0 < \overset{\circ}{U}_{\min}$$

is satisfied, then the inequality

$$0 < U < \overset{\circ}{U}_{\min}, \qquad T < \overset{\circ}{U}_{\min}$$

will be satisfied for all times during the motion, as will the inequalities

$$r < H, \qquad T < \overset{\circ}{U}_{\min}, \qquad \Delta < \delta_k$$

which are consequences of them.

This means that if the cavity and the function W are such that the position of the liquid completely filling the region D_k makes the potential energy of the liquid a minimum with respect to all possible displacements of the liquid surface, and if $U_{\min}$ is a positive-definite function of q_i, then the equilibrium position of the system will be stable with respect to r, T, Δ.

If the cavity has several simply connected volumes D_k° which are not interconnected, i.e. a particle may not pass from one volume into another without crossing the walls of the cavity, the definition and the proof are similar.

Let the body in the equilibrium position have two simply connected regions D_1°, D_2° corresponding to the same value $\alpha = \alpha_1 = \alpha_2$, and let the cavity be such that the liquid may be transferred within the cavity from one volume into the other.

We consider a possible position of the liquid, corresponding to some values of q_i, and where the liquid completely fills the regions D_1'' and D_2'', bounded at the "top" by the surface $\alpha + \Delta\alpha$, which is the same for both regions, and where this surface is obtained from the condition of conservation of the sum of the volumes, $V_1 + V_2$. If the empty portion of the cavity does not contain the points $W < \alpha + \Delta\alpha$, then in such a position of the liquid its potential energy will reach a minimum with respect to any displacements of the liquid particles which are possible for given q_i, and the condition of definiteness of the function $U_{\min}$ will be a sufficient condition for stability with respect to r, T, Δ.

Since we make use of the condition of conservation of the sum of the volumes, it is not difficult to see that the conditions of positive-definiteness of U_{min} are obtained the same as they would be if the regions D_1° and D_2° were connected by an infinitely thin channel, lying completely underneath the surface $W = \alpha$, i.e. if the problem were solved for a simply connected volume.

5. *Example 1.* The problem of Sretenskii [1]. We consider a certain generalization of the problem of Sretenskii. Let the constraints on the body be such that they allow only translational displacements of the body from the equilibrium position, and let its cavity be filled with a heavy liquid. If the potential energy of the frozen system has a minimum in the equilibrium position, then the equilibrium is stable. Actually, for any fixed displacements of the body the minimum potential energy of the liquid will be attained in the case where the surface of the liquid becomes horizontal and stationary with respect to the body. Consequently, in this case $U_2 \equiv 0$ and U will always be positive if U_1 proves to be positive. It is also not difficult to see that if the cavity is covered by a horizontal lid, if the walls of the cavity are vertical, and if the liquid occupies a single simply connected volume, then the stability with respect to the displacement r, T will follow from the positive-definiteness of U_1.

Example 2. A spherical pendulum containing a liquid.

We consider a heavy solid body with a fixed point O and a cavity filled with a heavy liquid. The fixed axis z is directed upwards. We choose as generalized coordinates the Euler angles φ', ψ', θ'. We denote by $-\ l(l > 0)$ the coordinate of the center of gravity of the frozen system in the equilibrium position, and by M its mass. It is convenient here to choose the coordinates x', y', z' in the moving system for the parameters ξ', η', ζ'; for the case of a heavy liquid

$$W = gz = g\,[z'\cos\theta' + \sin\theta'\,(x'\sin\varphi' + y'\cos\varphi')], \quad \frac{\partial W''}{\partial\theta'}\bigg|_{\theta'=0} = g\,(x'\sin\varphi' + y'\cos\varphi')$$

$$2\delta^2 U_2 = -\,\theta'^2\,(J' - md^2) = -\,a\,(\varphi)\,\theta'^2, \qquad J' = \sigma g \iint\limits_P (x'\sin\varphi' + y'\cos\varphi')^2\,dx'\,dy'$$

$$-\,md^2 = -\,\sigma g\,\frac{\lambda^2}{\nu}\,, \qquad \lambda = \iint\limits_P (x'\sin\varphi' + y'\cos\varphi')\,dx'\,dy', \qquad \nu = \iint\limits_P dx'\,dy'$$

The explanation of the notation introduced here follows: P is the surface consisting of the points of the frozen surface; J' is the moment of inertia of the surface P about the axis $x'\sin\varphi' + y'\cos\varphi' = 0$, where a unit area is assumed to have mass σg; $\sigma g\lambda$ is the first moment of the surface about the same axis; ν is the area of the surface, m is the mass of the surface, d the distance from the center of gravity of the surface to the axis $x' \sin\varphi' + y' \cos\varphi' = 0$, and $a(\varphi)$ the moment of inertia of this surface about the parallel axis through the center of gravity.

Let Ag be the maximum value of $a(\varphi)$. Then the equilibrium position of the system will be stable if

$$l - \frac{A}{M} > 0$$

If the cavity is a closed cylinder with generators parallel to the z-axis, then there will be stability with respect to θ, T, δ. This result was obtained by Moiseev [7] for a cavity having a rectangular surface P symmetrical about the projection of the axis y' on it.

Example 3. The stability of a pendulum filled with a liquid in gravitational and centrifugal force fields. We consider a right-hand rectangular coordinate system x, y, z with origin at the fixed point O and the vertical z-axis directed upwards, rotating about the z-axis with a constant angular velocity ω, and a heavy solid body able to rotate about the y-axis with respect to the x, y, z system. In the following we will understand motion to be motion with respect to this system. The cavity in the body is assumed to be filled with a homogeneous incompressible heavy liquid. The body and liquid will be in equilibrium if the center of gravity of the system lies on the z-axis at a distance l below the point of suspension, while the axis is a principal axis of inertia of the system at the point O. The free surface of the liquid will take the shape of the paraboloid

$$\frac{W}{g} = z - \frac{\omega^2}{2g}(x^2 + y^2) = z - \beta(x^2 + y^2) = -\alpha$$

We assume that the cavity has the form of a surface of revolution S about the z-axis and is intersected by the paraboloid along two circles lying in planes $z = -h$ and $z = -h - H$ with centers on the z-axis and radii d_1 and d_2 respectively:

$$d_1 > d_2, \qquad d_1^2\beta = \alpha - h, \qquad d_2^2\beta = \alpha - h - H$$

We recall also that the normals to the surfaces $W/g = -\alpha$ and S form a nonzero angle on the lines of intersection.

If O, x', y', z' is a moving coordinate system attached to the solid body and coinciding with O, x, y, z in the equilibrium position, and θ' is the angle of deviation of the z'-axis from the z-axis, then the formulas for the coordinate transformation are

$$x = x'\cos\theta' - z'\sin\theta', \qquad y = y', \qquad z = x'\sin\theta' + z'\cos\theta'$$

The functions in W/g are written in terms of these variables as

$$\frac{W}{g} = -z'^2\beta \sin^2\theta' + (\cos\theta' + \beta x' \sin 2\theta')\, z' - \beta x'^2 \cos^2\theta' - \beta y'^2 + x' \sin\theta'$$
$$\frac{1}{g}\frac{\partial W}{\partial\theta'} = -z'^2\beta \sin 2\theta' + (-\sin\theta' + 2\beta x' \cos 2\theta)\, z' - \beta x'^2 \sin 2\theta + x' \cos\theta'$$

Taking $\theta' = 0$, we obtain

$$\frac{1}{g}\frac{\partial W}{\partial\theta'}\bigg|_{\theta'=0} = 2\beta x'z' + x'$$

On the frozen surface $W/g = z' - \beta(x'^2 + y'^2) = -\alpha$, hence

$$\frac{1}{g}\frac{\partial W''^\circ}{\partial\theta'} = x' + 2\beta x'[\beta(x'^2 + y'^2) - \alpha]$$

In moving cylindrical coordinates z', $x' = \rho\cos\Psi$, $y' = \rho\sin\Psi$ we have

$$\frac{1}{g}\frac{\partial W''^\circ}{\partial\theta'} = \rho\cos\Psi + 2\beta\rho\cos\Psi\,(\beta\rho^2 - \alpha)$$

The region of integration on the plane z' is bounded by the circles $\rho = d_1$ and $\rho = d_2 (d_1 > d_2)$. In order to calculate $\delta^2 U_2$, as was shown above, it is necessary to calculate:

$$\lambda = \int_{d_2}^{d_1} \rho d\rho \int_0^{2\pi} d\Psi \frac{\partial W''^\circ}{\partial\theta'} = 0 \qquad (4.1)$$

$$-\delta^2 U_2 = \frac{\sigma g\theta'^2}{2}\int_{d_2}^{d_1} \rho d\rho \int_0^{2\pi} d\Psi\,[\rho\cos\Psi + 2\beta\rho\cos\Psi\,(\beta\rho^2 - \alpha)]^2 =$$
$$= \frac{\pi\sigma g\theta'^2}{2}\int_{d_2}^{d_1} [(1 - 2\alpha\beta)\,\rho + 2\beta^2\rho^3]^2 \rho d\rho =$$
$$= \frac{\pi\sigma g\theta'^2}{2}\left[\frac{(1-2\alpha\beta)^2 (d_1^4 - d_2^4)}{4} + \frac{2\beta^2(1-2\alpha\beta)(d_1^6 - d_2^6)}{3} + \frac{\beta^4(d_1^8 - d_2^8)}{2}\right]$$

We now assume that the body is homogeneous and symmetrical about the z'-axis and the cavity is a circular cylinder with axis z', radius R and height H, with walls at $z' = -h$, $z' = -h - H$. Let the volume of the liquid in this cylinder equal εV, where $V = \pi R^2 H$ is the volume of the cylinder, and ε is the coefficient of fullness.

Let the angular velocity ω of the body be increased, starting from zero. The paraboloid initially will be intersected by the lateral surface of the cylinder. Then, depending upon ε, R, and H, it may be intersected by the lateral surface and the bottom end or by only the top end, and then for ω sufficiently large it will be intersected by both the top and bottom ends. We consider the last case, for which

$$\beta R^2 + h > \alpha > h + H$$

Let $\alpha_1 = \alpha - h - H$ be the distance from the vertex of the paraboloid to the bottom end of the cylinder. Then the previous inequality takes the form $\beta R^2 - H > \alpha_1 > 0$. From the condition of constancy of volume of the liquid we find

$$\alpha_1 = \beta R^2 (1 - \varepsilon) - \frac{H}{2} \tag{4.2}$$

This is correct if β is larger than the greatest of the numbers $H/2R^2\varepsilon$ and $H/2R^2(1 - \varepsilon)$.

Under these conditions formula (4.1) is simplified to

$$\delta^2 U_2 = -\frac{\pi\sigma g\theta'^2}{2}\int_{d_2}^{d_1} \rho^2 [1 + 2\beta(\beta\rho^2 - \alpha)]^2 \rho d\rho = -\frac{\pi\sigma g\theta'^2}{16\beta^4}\int_{1-2\beta(H+h)}^{1-2\beta h} [U - (1 - 2\alpha\beta)]\, u^2 du$$

$$u = 1 + 2\beta(\beta\rho^2 - \alpha)$$

Denoting $h_1 = h + H/2$, integrating and making use of (4.2), we obtain

$$\delta^2 U_2 = -\theta'^2 \pi\sigma g H \left[\frac{H^2}{12\beta} - \frac{h_1 H^2}{6} + \frac{R^2(1-\varepsilon)}{4\beta} - R^2(1-\varepsilon) h_1 + \right.$$
$$\left. + R^2(1-\varepsilon) h_1^2 \beta + \frac{R^2(1-\varepsilon)\beta H^2}{12}\right] \tag{4.3}$$

The potential energy of the frozen system is

$$U_1 = (m_1 l_1 + m_2 l_2) g (1 - \cos\theta') - (B + B_1 - J_z)\beta g$$

$$J_z = (B + B_1)\cos^2\theta' + (C + C_1)\sin^2\theta'$$

$$g m_2 l_2 = 6\pi g H \left[R^2 \varepsilon h_1 + \frac{H^2}{12\beta}\right], \qquad B_1 = \sigma\pi H \left[\frac{R^4(2\varepsilon - \varepsilon^2)}{2} - \frac{H^2}{24\beta^2}\right]$$

$$C_1 = \sigma\pi H \left[\frac{R^4(2\varepsilon - \varepsilon^2)}{4} - \frac{H^2}{48\beta^2} + R^2\varepsilon h_1^2 + \frac{h_1 H^2}{6\beta} + \frac{R^2 \varepsilon H^2}{12}\right]$$

Here m_1, m_2 are the masses of the body and the liquid, l_1, l_2 are the distances of the centers of gravity of the heavy body and the frozen liquid from the point of suspension, B_1, B_2 are the moments of inertia of the body and the frozen liquid about the z-axis, and C, C_1 are the moments of inertia of the body and the frozen liquid about the y'-axis. For the second variation of U_1 we obtain

$$\delta^2 U_1 = \left[\frac{m_1 l_1 g}{2} + (B - C)\beta g\right]\theta'^2 + \sigma\pi g H \left[\frac{R^2 \varepsilon h_1}{2} + \frac{H^2}{24\beta} + \frac{\beta R^4(2\varepsilon - \varepsilon^2)}{4} - \right.$$
$$\left. - \frac{H^2}{48\beta} - R^2 \varepsilon h_1^2 \beta - \frac{h_1 H^2}{6} - \frac{R^2 \varepsilon H^2 \beta}{12}\right]\theta'^2 \tag{4.4}$$

As a result we obtain

$$\delta^2 U_{\min} = \delta^2 U_1 + \delta^2 U_2 = \frac{1}{2}(m_1 l_1 g + (B - C)\omega^2)\theta'^2 +$$
$$+ \frac{m_2}{\varepsilon}\left[\left(1 - \frac{\varepsilon}{2}\right)h_1 g + \frac{\omega^2}{2}\left(\frac{R^2(2\varepsilon - \varepsilon^2)}{4} - h_1^2 - \frac{H^2}{12}\right) - \frac{(1-\varepsilon)g^2}{2\omega^2}\right]\theta'^2$$

In order that the condition of positive-definiteness of $\delta^2 U$ will not be violated as ω increases, it is sufficient that

$$B - C + \frac{m_2}{\varepsilon}\left(\frac{R^2(2\varepsilon - \varepsilon^2)}{4} - h_1^2 - \frac{H^2}{12}\right) > 0$$

It is not difficult to see that in the given case the positive-definiteness of $\delta^2 U$ guarantees stability with respect to θ', T, Δ.

BIBLIOGRAPHY

1. Sretenskii, L.N., Kolebanie zhidkosti v podvizhnom sosude (Oscillations of a liquid in a moving container). *Izv. Akad. Nauk SSSR, OTN* No. 10. 1951.

2. Moiseev, N.N., Dvizhenie tela, imeiushchego polosti, chastichno zapolnennye ideal'noi kapel'noi zhidkost'iu (The motion of a body having a cavity partially filled with an ideal dropping liquid). *Dokl. Akad. Nauk SSSR* Vol. 20, No. 4, 1952.

3. Moiseev, N.N., Zadacha o dvizhenii tverdogo tela, soderzhashchego zhidkie massy, imeiushchie svobodnuiu poverkhnost' (The problem of motion of a solid body containing a mass of liquid having a free surface). *Mat. sb.* Vol. 32 (74), No. 1, 1953.

4. Narimanov, G.S., O dvizhenii tverdogo tela, polost' kotorogo chastichno zapolnena zhidkost'iu (On the motion of a solid body with a cavity partially filled with a liquid). *PMM* Vol. 20, No. 1, 1956.

5. Okhotsimskii, D.E., K teorii dvizheniia tela s polostiami, chastichno zapolnennymi zhidkost'iu (On the theory of the motion of a body with cavities partially filled with a liquid). *PMM* Vol.20, No.1, 1956.

6. Krein, S.G. and Moiseev, N.N., O kolebaniiakh tverdogo tela, soderzhashchego zhidkost' so svobodnoi poverkhnost'iu (On the oscillations of a solid body containing a liquid having a free surface). *PMM* Vol. 21, No. 2, 1957.

7. Moiseev, N.N., O dvukh maiatnikakh s zhidkost'iu (On two pendulums filled with liquid). *PMM* Vol. 20, No. 6, 1952.

8. Liapunov, A.M., Ob ustoichivosti ellipsoidal'nykh form ravnovesiia vrashchaiushcheisia zhidkosti (On the stability of the ellipsoidal equilibrium configurations of a rotating liquid). *Collected works*. Vol. 3, Izd-vo Akad. Nauk SSSR.

9. Rumiantsev, V.V., Ob ustoichivosti ravnovesiia tverdogo tela, imeiushchego polosti, napolnennye zhidkost'iu (On the stability of equilibrium of a solid body having cavities filled with a liquid). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 2, 1959.

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